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Self-Similar Solutions of the Boltzmann Equation for Non-Maxwell Molecules

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We show that the method previously used by the authors to obtain self-similar, eternal solutions of the space-homogeneous Boltzmann equation for Maxwell molecules yields different results when extended to other power-law potentials (including hard spheres). In particular, self-similar solutions cease to exist for a positive time for hard potentials. In the case of soft potentials, the solutions exist for all potive times, but are not eternal.

KEY WORDS: Eternal solutions; Boltzmann equation; self-similar solutions.

Let $f(\mathbf{v}, t)$ be the one-particle distribution function (here $\mathbf{v} \in \mathbb{R}^3$ and $t \in \mathbb{R}$ denote the velocity and time variables, respectively) of a spatially homogeneous system of gas molecules. Then the spatially homogeneous Boltzmann Equation (BE) reads as follows

$$\frac{\partial f}{\partial t} = Q(f, f) = \int_{\mathbb{R}^3} \int_{S^2} g\left(|\mathbf{u}|, \frac{\mathbf{u} \cdot \mathbf{n}}{|\mathbf{u}|}\right) [f(t, \mathbf{v}')\, f(t, \mathbf{w}') - f(t, \mathbf{v})\, f(t, \mathbf{w})]\, d\mathbf{n}\, d\mathbf{w} \tag{1}$$

where

$$\mathbf{u} = \mathbf{v} - \mathbf{w}, \qquad \mathbf{n} \in S^2, \qquad \mathbf{v}' = \tfrac{1}{2}(\mathbf{v} + \mathbf{w} + |\mathbf{u}|\, \mathbf{n}), \qquad \mathbf{w}' = \tfrac{1}{2}(\mathbf{v} + \mathbf{w} - |\mathbf{u}|\, \mathbf{n})$$

$$g(\mathbf{u}, \cos\theta) = |\mathbf{u}|\, \sigma(|\mathbf{u}|, \theta), \qquad 0 \leqslant \theta \leqslant \pi$$

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Here $\sigma(|\mathbf{u}|, \theta)$ is the differential cross-section at the scattering angle θ. We study the case of power-like potentials $U(r) = \alpha/r^n$ ($\alpha = 0$, $n > 1$) with or without angular cutoff. Then

$$g(|\mathbf{u}|) = |\mathbf{u}|^\gamma \, g_\gamma(\cos\theta), \qquad \gamma = 1 - \frac{4}{n} \tag{2}$$

The asymptotic case ($n \to \infty$)

$$g(|\mathbf{u}|) = |\mathbf{u}| \, \frac{d^2}{4} \tag{3}$$

corresponds to hard spheres with diameter d.

We assume that the initial distribution function $f_0(\mathbf{v}) \geqslant 0$ does not posses a second moment (energy) finite. Then we have:

$$\int_{\mathbb{R}^3} d\mathbf{v} \, f(\mathbf{v}, t)\{1, \mathbf{v}, |\mathbf{v}|^2\} = \{1, 0, \infty\} \tag{4}$$

provided these conditions are fullfilled at $t = 0$. The following proposition was proved in ref. 1.

Proposition. Equation (1) for Maxwell molecules ($\gamma = 0$ in (2)) has, for any $\lambda > 0$, non-negative self-similar solutions

$$f(\mathbf{v}, t) = e^{-3\lambda t} F(|\mathbf{v}| \, e^{-\lambda t}); \tag{5}$$

there are certain classes of solutions (satisfying (4) at $t = 0$) which are asymptotic to solutions (5) as $t \to \infty$. The solutions (5) are obviously eternal (valid for all $t \in \mathbb{R}$).

The aim of this note is to discuss briefly the following

Question. Can the above proposition be generalized to the case $\gamma \neq 0$, i.e., to non-Maxwell power potentials?

Following the same scheme as in ref. 1, we consider Eqs. (1) and (2) with arbitrary $\gamma \neq 0$ and transform $f(\mathbf{v}, t)$ to a scaling form

$$f(\mathbf{v}, t) = s^3(t) \, F[\mathbf{v}s(t), t] \tag{6}$$

with an unknown scaling function $s(t)$. Assuming that $s(t) > 0$, $F(\mathbf{v}, t) \geqslant 0$, we obtain the following equation for $F(\mathbf{v}, t)$:

$$s^\gamma \frac{\partial F}{\partial t} + (s^{\gamma - 1} s_t) \operatorname{div}_\mathbf{v}(\mathbf{v}F) = Q_\gamma(F, F) \tag{7}$$

where $Q_\gamma(F, F) = Q(F, F)$ with the collision kernel (2). The self-similar solution

$$f_*(\mathbf{v}, t) = s^3(t)\, F_*[\mathbf{v}s(t)] \tag{8}$$

leads to the equation:

$$s^{\gamma-1} s_t = -\lambda \Rightarrow s^\gamma = s^\gamma(0) - \lambda\gamma t \tag{9}$$

Moreover $\lambda\gamma > 0$ since

$$H(f_*) = \int_{\mathbb{R}^3} d\mathbf{v}\, f_*(\mathbf{v}, t) \log f_*(\mathbf{v}, t) = \int_{\mathbb{R}^3} d\mathbf{v}\, F_*(\mathbf{v}) \log F_*(\mathbf{v}) + 3 \log s(t) \tag{10}$$

must be a decreasing function of t.

Now we consider separately two cases :

(A) soft potentials with $\gamma < 0$;

(B) hard potentials with $\gamma > 0$.

In the first case ($\gamma < 0$) we obtain:

$$s(t) = s_0 (1 + \lambda\, |\gamma|\, s_0^{|\gamma|} t)^{-1/|\gamma|}, \qquad s_0 = s(0) \tag{11}$$

If we introduce the notation

$$\tau(t) = \int_0^t \frac{dt'}{s(t')} \tag{12}$$

then Eq. (7) for $\tilde{F}(\mathbf{v}, \tau) = F(\mathbf{v}, t)$ reads

$$\frac{\partial \tilde{F}}{\partial \tau} - \lambda\, \mathrm{div}_{\mathbf{v}}(\mathbf{v}\tilde{F}) = Q_\gamma(\tilde{F}, \tilde{F}) \tag{13}$$

Moreover $\tau(t) \to \infty$ with t. Equation (13) is very similar (in both cases of positive and negative values of γ) to that with $\gamma = 0$. Its physical meaning is obvious: it is the Boltzmann equation for a space-homogeneous, rarefied gas in a dissipative medium with a constant coefficient of friction λ. By an analogy with the case $\gamma = 0$ [(1)] we can expect that there exists a stationary solution $F_*(\mathbf{v})$ with infinite energy and that this solution is an attractor for certain classes of initial data. Hence, we conjecture that the above proposition, where Eq. (5) is replaced by Eqs. (8) and (11), is also valid for soft potentials (with the exception of the statement indicating that they are eternal, see later).

Let us now consider the case (B) of hard potentials ($\gamma > 0$). Then Eq. (9) results in

$$s(t) = s_0(1 - \lambda\gamma s_0^{-\gamma} t)^{1/\gamma}, \qquad s_0 = s(0), \quad 0 < \gamma \leqslant 1 \tag{14}$$

By using the same substitution (12) we obtain Eq. (13) again, and expect that there exists a stationary solution $F_*(\mathbf{v})$ and that $\tilde{F}(\mathbf{v}, \tau) \to F_*(\mathbf{v})$ as $\tau \to \infty$ for a certain class of initial data. There is, however, an essential difference from the case $\gamma \leqslant 0$: the limit $\tau \to \infty$ does not correspond to the limit $t \to \infty$, but to the finite value

$$t = t_*(\gamma) = \frac{s_0^\gamma}{\lambda\gamma}, \qquad \gamma > 0$$

On the other hand, $\tau(t) \to \infty$ as $t \to t_*(\gamma)$ for all $0 < \gamma \leqslant 1$. Thus we expect that for some initial data satisfying (4) the solution $f(\mathbf{v}, t)$ is defined for $0 < t < T_*(\gamma)$ and

$$f_*(\mathbf{v}, t) \cong_{t \to t_*} s_\gamma^3(t)\, F_*[\mathbf{v} s_\gamma(t)]$$

$$s_\gamma(t) = \text{const.}[t_*(\gamma) - t]^{1/\gamma}, \qquad 0,\ \gamma < 1$$

Then the collision frequency behaves asymptotically (as $t \to t_\gamma$) as

$$\int d\mathbf{w}\, f(\mathbf{v}, t)\, |\mathbf{w} - \mathbf{w}|^\gamma \cong \int d\mathbf{w}\, f(\mathbf{w}, t)\, |\mathbf{w}|^\gamma \cong \frac{1}{t_*(\gamma) - t}$$

and therefore the solution cannot be extended to larger values, $t \geqslant t_*(\gamma)$. This means that global in time solutions with infinite energy does not exist for power potentials of the hard type (in particular, rigid spheres). On the other hand, the possible self-similar solution (8) for $\gamma > 0$ is well defined for all $t < 0$, whereas, as we have seen, it becomes singular at a certain positive time. The solutions for soft-potentials exist for all positive times, but become singular for a certain negative time. Hence it seems probable that eternal solutions (defined for all $t \in \mathbb{R}$) exist only for Maxwell molecules (among other power-like potentials)

In spite of these results, a self-similar solution may still be used to describe the asymptotics of the solution describing the structure of an infinitely strong shock wave, in the manner indicated in ref. 1.

Of course, all the above considerations were made at a rather formal level. The results, however, were formulated as clear mathematical conjectures which can be proved or disproved rigorously. One of the goals of this note is to attract the attention of mathematicians to this interesting open problem.

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